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New Refinements of the Jensen Inequalities Based on Samples with Repetitions*

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In this paper we give a refinement of the Jensen inequality for any mid-convex function f by arithmetic means of f evaluated at arithmetic means of samples with repetitions. The case of samples without repetitions was treated by J. E. Pečarić and V. Volenec (*Österreich. Akad. Wiss. Math.-Natur. Kl. Sonderdruck Sitzungsber.* **197**, 1988, 463–467).

1. INTRODUCTION

Let I be a convex subset of an arbitrary real linear space X. A function $f : I \to \mathbf{R}$ is called mid-convex if for every two elements $x, y \in I$ we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{1}$$

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It is well known that for every mid-convex function f the Jensen inequality

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i), \tag{2}$$

where $x_i \in I$ $(i = 1, \dots, n)$, is valid.

Set

$$f_{k,n} = f_{k,n}(x_1, \dots, x_n) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right) \tag{3}$$

$$\begin{aligned}\bar{f}_{k,n} &= \bar{f}_{k,n}(x_1, \dots, x_n) \\ &= \frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le \cdots \le i_k \le n} f\left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right).\end{aligned} \tag{4}$$

In Pečarić and Volenec [3] the following finite refinement of the Jensen inequality (2) is proved,

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = f_{n,n} \le \cdots \le f_{k+1,n} \le f_{k,n} \le \cdots \le f_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{5}$$

(see also Gabler [4], Mitrinović and Pečarić [2], and Pečarić [1]).

2. MAIN RESULTS

In this paper we give the following (infinite!) refinement of Jensen's inequality (2):

THEOREM 1. *If $f: I \to \mathbf{R}$ is a mid-convex function, and $x_i \in I$, $i = 1, \dots, n$, then for all $k = 1, 2, \dots$ the following refinement*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \cdots \le \bar{f}_{k+1,n} \le \bar{f}_{k,n} \le \cdots \le \bar{f}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{6}$$

of the Jensen inequality holds true.

First we prove the following

KEY LEMMA. *Let f be mid-convex. Then*

$$f\left(\frac{1}{n}\sum_{j=1}^{n} y_j\right) \le \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{y_1 + \cdots + \hat{y}_j + \cdots + y_n}{n-1}\right), \tag{7}$$

where $\hat{y}_j$ means that y_j is omitted.

Proof. Apply (2) to $y^{(i)} := (1/(n-1))(y_1 + \cdots + \hat{y}_i + \cdots + y_n)$ and use the (obvious) identity $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y^{(i)}$.

Proof of Theorem 1. To prove that $\bar{f}_{k+1,n} \le \bar{f}_{k,n}$ we first apply the Key Lemma to every term in the sum

$$\sum_{1 \le i_1 \le \cdots \le i_{k+1} \le n} f\left(\frac{x_{i_1} + \cdots + x_{i_{k+1}}}{k+1}\right) \qquad \left(= \binom{n+k}{k+1}\bar{f}_{k+1,n}\right)$$

which makes it

$$\le \frac{1}{k+1} \sum_{1 \le i_1 \le \cdots \le i_{k+1} \le n} \sum_{j=1}^{k+1} f\left(\frac{x_{i_1} + \cdots + \hat{x}_{i_j} + \cdots + x_{i_{k+1}}}{k}\right).$$

This last expression, being symmetric in the x_i's, can be rewritten as

$$\frac{1}{k+1} c_{n,k} \sum_{1 \le i'_1 \le \cdots \le i'_k \le n} f\left(\frac{x_{i'_1} + \cdots + x_{i'_k}}{k}\right),$$

where the constant $c_{n,k}$ is equal to the number of $1 \le i_1 \le \cdots \le i_{k+1} \le n$ such that $\{i_1 \le \cdots \le \hat{i}_j \le \cdots \le i_{k+1}\} = \{i'_1 \le \cdots \le i'_k\}$ for some j. If $\{i'_1 \le \cdots \le i'_k\} = \{1^{r_1} 2^{r_2} \cdots n^{r_n}\}$, then it is clear that

$$c_{n,k} = (r_1 + 1) + (r_2 + 1) + \cdots + (r_n + 1) = k + n.$$

All together we get

$$\binom{n+k}{k+1}\bar{f}_{k+1,n} \le \frac{n+k}{k+1}\binom{n+k-1}{k}\bar{f}_{k,n}, \qquad \text{i.e., } \bar{f}_{k+1,n} \le \bar{f}_{k,n}.$$

To complete the proof of (6) we only need to prove that

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \bar{f}_{k,n} \qquad (k \ge 1).$$

Indeed, we have

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = f\left(\frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le \cdots \le i_k \le n} \left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right)\right)$$

$$\le \frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le \cdots \le i_k \le n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) = \bar{f}_{k,n}.$$

Remark. Note that our Key Lemma is just the other way of writing the inequality $f_{n,n} \le f_{n-1,n}$ from Pečarić and Volenec [3], and note that one obtains the other inequalities $f_{k+1,n} \le f_{k,n}$ in (3) simply by iterating it.

Now we study some inequalities relating the functions $f_{k,n}$ and $\bar{f}_{k,n}$. So far we can prove the following

THEOREM 2. *For any mid-convex function* $f : I \to \mathbf{R}$ *and its associated functions* $f_{k,n}$, $\bar{f}_{k,n}$ *as defined in* (3), (4) *the following inequalities*

$$\bar{f}_{k,n} \ge \sum_{s=1}^{\min\{k,n\}} \frac{\binom{k-1}{s-1}\binom{n}{s}}{\binom{n+k-1}{k}} f_{s,n} \tag{8}$$

$$\bar{f}_{k,n} \ge f_{\min\{k,n\},n} \tag{9}$$

hold true.

Proof. If we write $\binom{n+k-1}{k}\tilde{f}_{k,n}$ in the (equivalent) form

$$\binom{n+k-1}{k}\bar{f}_{k,n}$$

$$= \sum_{s=1}^{\min\{k,n\}} \sum_{1 \le j_1 < \cdots < j_s \le n} \left[\sum_{\substack{p_1 + \cdots p_s = k \\ p_1 > 0, \ldots, p_s > 0}} f\left(\frac{p_1 x_{j_1} + \cdots + p_s x_{j_s}}{k}\right)\right]$$

and use the Jensen inequality for the $\binom{k-1}{s-1}$ points $(p_1 x_{j_1} + \cdots + p_s x_{j_s})/k$ (j_r's fixed) corresponding to compositions (i.e., ordered partitions) $p_1 + \cdots + p_s = k$ of k into s (nonzero) parts, we get

$$\binom{n+k-1}{k}\bar{f}_{k,n} \ge \sum_{s=1}^{\min\{k,n\}} \binom{k-1}{s-1} \sum_{1 \le j_1 < \cdots < j_s \le n} f\left(\frac{x_{j_1} + \cdots + x_{j_s}}{s}\right)$$

$$= \sum_{s=1}^{\min\{k,n\}} \binom{k-1}{s-1}\binom{n}{s} f_{s,n} \qquad \text{by (3)}.$$

Hence, the inequality (8) follows.

The inequality (9) follows simply from (8) by using (5) in the form $f_{s,n} \geq f_{k,n}$ $(s \leq k)$ and the identity (Vandermonde's convolution)

$$\sum_{s=1}^{k} \binom{k-1}{s-1}\binom{n}{s} = \binom{n+k-1}{k}.$$

Let us summarize our inequalities in the following diagram (a ladder of inequalities!)

$$\begin{tikzcd}
\bar{f}_{\infty,n} \arrow[d, equal] & \le \cdots \le & \bar{f}_{n,n} & \le \cdots \le & \bar{f}_{\alpha(k),n} \arrow[dr, "?"] & \le \cdots \le & \bar{f}_{k,n} \arrow[dr, "?"] & \le \cdots\cdots\cdots \le & \bar{f}_{1,n} \arrow[d, equal] \\
f_{n,n} \arrow[urr] & \le \cdots\cdots\cdots\cdots & & & \le & f_{k,n} \arrow[ur] & \le \cdots \le & f_{\beta(k),n} \le \cdots & f_{1,n}
\end{tikzcd}$$

from which it is natural to pose the following

Problem. Find (minimal) $\alpha(k) \leq \infty$, and (maximal) $\beta(k)$ such that

$$\bar{f}_{\alpha(k),n} \leq f_{k,n} \leq \bar{f}_{k,n} \leq f_{\beta(k),n}.$$

Of course, we know that $k \leq \alpha(k) \leq \infty$ and $1 \leq \beta(k) \leq k$.

3. GENERALISATIONS

Now let $\mathbf{R}^l$ denote the l-dimensional vector lattice of points $\mathbf{x} = (x_1, \ldots, x_l)$, x_i real for $i = 1, \ldots, l$, with the partial ordering

$$\mathbf{x} = (x_1, \ldots, x_l) \leq \mathbf{y} = (y_1, \ldots, y_l) \tag{10}$$

if and only if $x_i \leq y_i$ for $i = 1, \ldots, l$. We shall write $ax + by = (ax_1 + by_1, \ldots, ax_l + by_l)$, where $a, b \in \mathbf{R}$.

A real valued function f on an interval $I \subset \mathbf{R}^l$ will be said to have nondecreasing increments if

$$f(a+h) - f(a) \leq f(b+h) - f(b) \tag{11}$$

whenever $a \in I$, $b + h \in I$, $0 = (0, \ldots, 0) \leq h \in \mathbf{R}^l$, $a \leq b$.

Note that functions $f(x, y) = xy$ $(x, y \in \mathbf{R})$ and $f(x_1, \ldots, x_l) = x_1 x_2 \ldots x_l$ $(x_i \geq 0,\ j = 1, \ldots, l)$ have nondecreasing increments.

If we put $a = x$, $b + h = y$, $b = a + h$, then (11) becomes (1) with $x \leq y$. Note that if $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \cdots \leq \mathbf{x}_n$ $(\mathbf{x}_i \in \mathbf{R}^l)$ we have that (2) is valid (Pečarić [1]).

Now, we shall give the following refinement of this result.

THEOREM 3. *If* $f: I \to \mathbf{R}$, $I =$ *interval in* $\mathbf{R}^l$, *is a function with nondecreasing increments and if* $\mathbf{x}_i \in I$, $i = 1, \ldots, n$, *then we have*

$$\bar{f}_{k+1,n} \leq \bar{f}_{k,u}, \qquad k = 1, 2, \ldots .$$

Proof. Since from $x_{i_1} \leq \cdots \leq x_{i_{k+1}}$, it follows that

$$\frac{1}{k}\left(x_{i_1} + \cdots + \hat{x}_{i_{k+1}}\right) \leq \cdots \leq \frac{1}{k}\left(\hat{x}_{i_1} + \cdots + x_{i_{k+1}}\right)$$

and we can work as in the proof of Theorem 1.

Remark 1. Note that the result given in Theorem 3 is also valid for a wider class of functions, i.e., for functions that satisfy (1) for all x, y chosen from I such that $x \leq y$.

4. APPLICATIONS

It is well known that by specializing Jensen's inequality one gets many other well-known inequalities. Let us, for the reader's convenience, state two special refinements of the arithmetic-geometric mean inequalities, in slightly better notations (than in the literature) which correspond to the following choices of the function f.

Case 1. $f(x) = -\log x$. We have

$$\frac{1}{n}\sum_{j=1}^{n} x_j \geq \cdots \geq (GA)_{k+1,n} \geq (GA)_{k,n} \geq \cdots \geq \sqrt[n]{x_1 \cdots x_n} \qquad (1 \leq k \leq n), \quad (5')$$

where

$$(GA)_{k,n} := \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right)^{1/\binom{n}{k}}$$

is the geometric mean of the arithmetic means of k-samples (out of n) without repetitions.

$$\frac{1}{n}\sum x_j \geq \cdots \geq (\overline{GA})_{k+1,n} \geq (\overline{GA})_{k,n} \geq \cdots \geq \sqrt[n]{x_1 \cdots x_n} \qquad (1 \leq k < \infty), \quad (6')$$

where

$$(\overline{GA})_{k,n} := \prod_{1\le i_1\le \cdots \le i_k\le n} \left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right)^{1/\binom{n+k-1}{k}}$$

is the geometric mean of the arithmetic means of k-samples (out of n) with repetitions.

$$(\overline{GA})_{k,n} \le \prod_{s=1}^{\min\{k,n\}} (GA)_{s,n}^{\binom{k-1}{s-1}\binom{n}{s}/\binom{n+k-1}{k}} \tag{8$'$}$$

$$(\overline{GA})_{k,n} \le (GA)_{\min\{k,n\},n}. \tag{9$'$}$$

Case 2. $f(x) = e^x$, $x_i \to \ln x_i$ $(i = 1, \ldots, n)$. We have

$$\frac{1}{n}\sum_{i=1}^{n} x_i \ge \cdots \ge (AG)_{k,n} \ge (AG)_{k+1,n} \ge \cdots \ge \sqrt[n]{x_1 \cdots x_n}$$

$$(1 \le k \le n), \tag{5$''$}$$

where

$$(AG)_{k,n} := \left(\sum_{1\le i_1< \cdots <i_k\le n} \sqrt[k]{x_{i_1} \cdots x_{i_k}}\right) \Bigg/ \binom{n}{k}$$

is the arithmetic mean of the geometric means of k-samples (out of n) without repetitions.

$$\frac{1}{n}\sum x_i \ge \cdots \ge (\overline{AG})_{k,n} \ge (\overline{AG})_{k+1,n} \ge \cdots \ge \sqrt[n]{x_1 \cdots x_n}$$

$$(1 \le k < \infty), \tag{6$''$}$$

where

$$(\overline{AG})_{k,n} := \left(\sum_{1\le i_1\le \cdots \le i_k\le n} \sqrt[k]{x_{i_1} \cdots x_{i_k}}\right) \Bigg/ \binom{n+k-1}{k}$$

is the arithmetic mean of the geometric means of k-samples (out of n) with repetitions.

$$(\overline{AG})_{k,n} \le \sum_{s=1}^{\min\{k,n\}} \frac{\binom{k-1}{s-1}\binom{n}{s}}{\binom{n+k-1}{k}} (AG)_{s,n} \tag{8$''$}$$

$$(\overline{AG})_{k,n} \ge (AG)_{\min\{k,n\},n}. \tag{9$''$}$$

Remark 2. Let us denote by

$$E_k(x_1,\dots,x_n) = \binom{n}{k}^{-1} \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k} \qquad (1\le k\le n)$$

$$H_k(x_1,\dots,x_n) = \binom{n+k-1}{k}^{-1} \sum_{1\le i_1\le\cdots\le i_k\le n} x_{i_1}\cdots x_{i_k} \qquad (k\le 1)$$

the normalized elementary, resp. complete homogeneous, symmetric functions. Then the inequalities (5″)–(9″) can be rewritten as

$$E_1(x_1,\dots,x_n) \ge \cdots \ge E_k\left(x_1^{1/k},\dots,x_n^{1/k}\right) \ge \cdots \ge E_n\left(x_1^{1/n},\dots,x_n^{1/n}\right) \tag{12}$$

$$H_1(x_1,\dots,x_n) \ge \cdots \ge H_k\left(x_1^{1/k},\dots,x_n^{1/k}\right) \ge \cdots \ge H_\infty\left(x_1^{1/n},\dots,x_n^{1/n}\right), \tag{13}$$

where $H_\infty(x_1^{1/n},\dots,x_n^{1/n}) := \sqrt[n]{x_1\cdots x_n}$.

$$H_k\left(x_1^{1/k},\dots,x_n^{1/k}\right) \ge \sum_{s=1}^{\min\{k,n\}} \frac{\binom{k-1}{s-1}\binom{n}{s}}{\binom{n+k-1}{k}} E_s\left(x_1^{1/s},\dots,x_n^{1/s}\right) \tag{14}$$

$$H_k\left(x_1^{1/k},\dots,x_n^{1/k}\right) \ge E_{k'}\left(x_1^{1/k'},\dots,x_n^{1/k'}\right), \qquad \text{where } k' = \min\{k,n\}. \tag{15}$$

The inequalities (12) and (13) seem to be interesting companions to the well-known Maclaurin inequalities which read as

$$E_1(x_1,\dots,x_n) \ge \left(E_2(x_1,\dots,x_n)\right)^{1/2} \ge \cdots \ge E_n(x_1,\dots,x_n)^{1/n}$$

and to the following inequalities (see [5, p. 164]):

$$H_1(x_1,\dots,x_n) \le \left(H_2(x_1,\dots,x_n)\right)^{1/2} \le \left(H_3(x_1,\dots,x_n)\right)^{1/3} \le \cdots.$$

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